Sinusoidal Functions as Mathematical Models

1. **Bouncing Spring Problem:** A weight attached to the end of a long spring is bouncing up and down. As it bounces, its distance from the floor varies sinusoidally with time. You start a stopwatch. When the stopwatch reads 0.3 seconds, the weight first reaches a high point of 60 cm. above the floor. The next low point, 40 cm. above the floor, occurs at 1.8 seconds.

a) Sketch a graph of this sinusoidal function.

\[ y = 10 \cos \left[ \frac{2\pi}{3} (t - 0.3) \right] + 50 \]

b) Write an equation expressing distance from the floor in terms of the number of seconds the stopwatch reads.

\[ r = 3.3 - 0.3 t \]

\[ \theta = \frac{2\pi}{3} \]

\[ \varphi = 0.3 \]

c) Predict the distance from the floor when you started the stopwatch.

\[ y = 10 \cos \left[ \frac{2\pi}{3} (0 - 0.3) \right] + 50 = 58.090 \text{ cm} \]

d) Predict the first positive value of time at which the weight is 59 cm. above the floor.

\[ 59 = 10 \cos \left[ \frac{2\pi}{3} (t - 0.3) \right] + 50 \]

\[ 9 = 10 \cos \left[ \frac{2\pi}{3} (t - 0.3) \right] \]

\[ y_{10} = 10 \cos \left[ \frac{2\pi}{3} (t - 0.3) \right] \]

\[ \cos^{-1} \left( \frac{9}{10} \right) = \frac{2\pi}{3} (t - 0.3) \]

\[ t = 0.085 \text{ seconds} \]

e) How high is the weight above the floor after 3 minutes? Is it going up or down? How can you tell?

\[ 3(60) = 180 \text{ seconds} \]

\[ y = 10 \cos \left[ \frac{2\pi}{3} (180 - 0.3) \right] + 50 \]

\[ y = 58.090 \text{ cm} \]

180 seconds is 59 periods after 3 seconds.

From the graph, it is clearly moving up at 3 seconds, therefore it is also moving up at 180 seconds.
2. **Tarzan Problem:** Tarzan is swinging back and forth on his grapevine. As he swings, he goes back and forth across the riverbank, going alternately over land and water. Jane decides to model mathematically his motion and starts a stopwatch. Let \( t \) be the number of seconds the stopwatch reads and let \( y \) be the number of meters Tarzan is from the riverbank. Assume that \( y \) varies sinusoidally with \( t \), and that \( y \) is positive when Tarzan is over water and negative when he is over land. Jane finds that when \( t = 2 \), Tarzan is at one end of his swing, where \( y = -23 \). She finds that when \( t = 5 \) he reaches the other end of his swing and \( y = 17 \).

a) Sketch a graph of this sinusoidal function.

![Graph of a sinusoidal function](image)

b) Write an equation expressing Tarzan's distance from the riverbank in terms of \( t \).

\[
y = 20 \cos \left( \frac{\pi}{3} (t - 5) \right) - 3
\]

c) Predict \( y \) when \( t = 2.8 \), \( t = 6.3 \), and \( t = 15 \).

\[
y = 20 \cos \left( \frac{\pi}{3} (2.8 - 5) \right) - 3 \quad y = 20 \cos \left( \frac{\pi}{3} (6.3 - 5) \right) - 2 \quad y = 20 \cos \left( \frac{\pi}{3} (15 - 5) \right) - 3
\]

\[
y = -16.383 \quad y = 1.158 \quad y = -13
\]

d) Where was Tarzan when Jane started the stopwatch?

\[
y = 20 \cos \left( \frac{\pi}{3} (0 - 5) \right) - 3 \quad y = 7
\]

e) Find the least positive value of \( t \) for which Tarzan is directly over the riverbank.

\[
1.920 = \frac{\pi}{3} (t - 5) \quad -1.920 = \frac{\pi}{3} (t - 5)
\]

\[
1.356 = t - 5 \quad -1.356 = t - 5
\]

\[
t = 6.356 + 6n \quad t = 3.644
\]

\[
\cos \frac{\pi}{20} = \frac{\pi}{3} (t - 5)
\]

\[
\cos \frac{\pi}{20} = 0.356 \text{ seconds}
\]
3. **Roller Coaster Problem:** A portion of a roller coaster track is to be built in the shape of a sinusoid (since it is a roller coaster, the actual ride starts at the top). You have been hired to calculate the lengths of the horizontal and vertical timber supports to be used. The high and low points are separated by 50 meters horizontally and by 30 meters vertically. The low point is 3 meters below the ground.

a) Sketch a graph of this sinusoidal function.

![Graph of a sinusoidal function]

b) Letting \( y \) be the number of meters the track is above the ground and \( x \) be the number of meters horizontally from the high point, write an equation expressing \( y \) in terms of \( x \).

\[
y = 15 \cos \left( \frac{\pi}{50} x \right) + 12
\]

![Equation for \( y \) in terms of \( x \)]

c) How long is the vertical timber at the high point, at \( x = 4 \) meters, and at \( x = 32 \) meters?

\[
y = 15 \cos \left( \frac{\pi}{50} (4) \right) + 12
\]

\[
y = 15 \cos \left( \frac{\pi}{50} (32) \right) + 12
\]

\[
y = 26.569 \text{ meters}
\]

\[
y = 5.613 \text{ meters}
\]

d) How long are the horizontal timbers that are 25 meters above the ground and 5 meters above the ground?

\[
25 = 15 \cos \left( \frac{\pi}{50} x \right) + 12
\]

\[
13 = 15 \cos \left( \frac{\pi}{50} x \right)
\]

\[
\frac{12}{15} = \cos \left( \frac{\pi}{50} x \right)
\]

\[
0.8 = \cos \left( \frac{\pi}{50} x \right)
\]

\[
x = 8.313 + 100n
\]

\[
x = -8.313 + 100n
\]

\[
x = 91.687 + 100n
\]

\[
x = 8.313 \text{ meters}
\]
4. **Buried Treasure Problem:** You seek a treasure that is buried in the side of a mountain. The mountain range has a sinusoidal cross-section. The valley to the left is filled with water to a depth of 50 meters, and the top of the range is 150 meters above sea level. You set an x-axis at water level and a y-axis 200 meters to the right of the deepest part of the water. The top of the mountain is at \( x = 400 \) meters.

a) Sketch a graph of this sinusoidal function.

\[ \gamma = 100 \cos \left( \frac{\pi}{600} (x - 400) \right) + 50 \]

b) Write an equation expressing \( y \) in terms of \( x \) for points on the surface of the mountain.

\[ y = 100 \cos \left[ \frac{\pi}{600} (x - 400) \right] + 50 \]

\[ \frac{\partial y}{\partial x} = \frac{\pi}{600} \cos \left[ \frac{\pi}{600} (x - 400) \right] \]

\[ \gamma = 0 \]

c) Show by calculation that the sinusoid contains the point \((0, 0)\).

\[ y = 100 \cos \left( \frac{\pi}{600} (0 - 400) \right) + 50 \]

\[ y = 0 \]

d) The treasure is located within the mountain at a point \((x, y) = (130, 40)\). This point is not on the graph. Which would be shorter way to dig to the treasure, a horizontal tunnel or a vertical tunnel? Justify your answer.

**Vertical:**

\[ y = 100 \cos \left( \frac{\pi}{600} (130 - 400) \right) + 50 \]

\[ y = 65.643 \]

\[ 65.643 - 40 = 25.643 \text{ meters} \]

**Horizontal:**

\[ y = 100 \cos \left( \frac{\pi}{600} (x - 400) \right) + 50 \]

\[ -10 = 100 \cos \left( \frac{\pi}{600} (x - 400) \right) \]

\[ \cos^{-1} \left( \frac{-10}{100} \right) = \frac{\pi}{600} (x - 400) \]

\[ x = 80.869 + 1200 n \]

A **vertical tunnel** would be shorter.

\[ 130 - 80.869 = 49.131 \text{ meters} \]
5. **Sunspots Problem**: You have kept track of the number of solar flares, or "sunspots," which occur on the surface of the Sun. The number of sunspots counted in a given year varies periodically from a minimum of about 10 per year to a maximum of about 110 per year. Between the maximums that occurred in the years 1750 to 1948, there were 18 complete cycles.

a) Assume that the number of sunspots counted in a year varies sinusoidally with the year. Sketch a graph of two sunspot cycles, starting in 1948.

![Graph of sunspots](image)

b) Write an equation expressing the number of sunspots per year in terms of the year. Use an appropriate value for the phase displacement.

\[ P = \frac{1948 - 1752}{18} \quad \beta = \frac{2\pi}{11} \]

\[ y = 50 \cos \left[ \frac{2\pi}{11} \left( t - 1948 \right) \right] + 60 \]

\[ P = 11 \]

c) How many sunspots would you expect in the year 2000?

\[ y = 50 \cos \left[ \frac{2\pi}{11} \left( 2000 - 1948 \right) \right] + 60 \]

\[ y = 52.38 \]

About 53

d) How many sunspots would you expect this year?

\[ y = 50 \cos \left[ \frac{2\pi}{11} \left( \text{this year} - 1948 \right) \right] + 60 \]

\[ y = \text{？”} \]

e) What is the first year after 2000 in which the number of sunspots will be about 35?

\[ 35 = 50 \cos \left[ \frac{2\pi}{11} \left( t - 1948 \right) \right] + 60 \]

\[ 2.05 \pi = \frac{2\pi}{11} \left( t - 1948 \right) \]

\[ 3.667 = t - 1948 \]

\[ -3.667 = t - 1948 \]

\[ t = 1951.667 + 11 \]

\[ t = 1951.667 + 11(5) \]

\[ t = 2006.667 \]

\[ \boxed{2006} \]

f) What is the first year after 2000 in which there will be a maximum number of sunspots?

\[ 1948 + 11(5) = \boxed{2003} \]
6. **Tide Problem:** Supposed that you are on the beach at Port Aransas, Texas. At 2:00 p.m. on March 19, the tide is in (i.e. the water is at its deepest). At that time you find that the depth of the water at the end of the pier is 1.5 meters. At 8:00 p.m. the same day when the tide is out, you find that the depth of the water is 1.1 meters. Assume that the depth of the water varies sinusoidally with time.

a) Sketch a graph of this sinusoidal function.

![Graph of a sinusoidal function]

\[ y = 0.2 \cos \left( \frac{\pi}{8} (t - 2) \right) + 1.3 \]

b) Derive an equation expressing the depth of the water in terms of the number of hours that have elapsed since 12:00 noon on March 19.

- Since the period is 12 hours, neither the date nor a.m. vs. p.m. matter.

\[ y = 0.2 \cos \left( \frac{\pi}{8} (t - 2) \right) + 1.3 \]

\( y = 1.3 \text{ meters} \)

c) Use the mathematical models to predict the depth of the water at 4:00 p.m. on March 19, 7:00 a.m. on March 20, and 5:00 p.m. on March 20.

\[ y = 0.2 \cos \left( \frac{\pi}{8} (t - 2) \right) + 1.3 \]

\( y = 1.37 \text{ meters} \)

d) At what time will the first low tide occur on March 20?

All low tides first occur at 8:00 a.m.

e) What is the earliest time on March 20 that the water will be 1.27 meters deep?

\[ 1.27 = 0.2 \cos \left( \frac{\pi}{8} (t - 2) \right) + 1.3 \]

\[ 1.721 = \frac{\pi}{8} (t - 2) \]

\[ -1.721 = \frac{\pi}{8} (t - 2) \]

\[ -0.288 = t - 2 \]

\[ 3.288 = t - 2 \]

\[ t = 5.288 + 12n \]

\[ t = -1.288 + 12n \]

\[ t = 10.712 \]

f) Calculate the first time the water is 2 meters deep on March 31.

The depth will never be 2 meters.
7. **Tidal Wave Problem:** A tsunami (commonly called a “tidal wave” because its effect is like a rapid change in tide) is a fast moving ocean wave caused by an underwater earthquake. The water first goes down from its normal level, then rises an equal distance above its normal level, and finally returns to its normal level. The period is about 15 minutes. Suppose that a tsunami with an amplitude of 10 meters approaches the pier at Honolulu, where the normal depth of the water is 9 meters.

a) Sketch a graph of this sinusoidal function.

b) Write an equation expressing the height of the wave as a function of time.

\[ y = 10 \cos \left( \frac{2\pi}{15} (t - 11.25) \right) + 9 \]

c) Predict the depth of the water after the tsunami first reaches the pier after 2 minutes, after 4 minutes, and after 12 minutes.

- \[ y = 10 \cos \left( \frac{2\pi}{15} (2 - 11.25) \right) + 9 \]  
  \[ y = 10 \cos \left( \frac{2\pi}{15} (-9) \right) + 9 \]  
  \[ y = 1.567 \text{ meters} \]

- \[ y = 10 \cos \left( \frac{2\pi}{15} (4 - 11.25) \right) + 9 \]  
  \[ y = 10 \cos \left( \frac{2\pi}{15} (-7.25) \right) + 9 \]  
  \[ y = -0.945 \text{ meters} \]

- \[ y = 10 \cos \left( \frac{2\pi}{15} (12 - 11.25) \right) + 9 \]  
  \[ y = 10 \cos \left( \frac{2\pi}{15} (1.25) \right) + 9 \]  
  \[ y = 18.51 \text{ meters} \]

d) According to your model, what will the minimum depth of the water be? How do you interpret this answer in terms of what will happen in the real world?

-1 meters. As the wave comes in, the water will first recede, and there will be a period of time with no water at the pier.

e) Between what two times is there no water at the pier?

\[ 0 = 10 \cos \left( \frac{2\pi}{15} (t - 11.25) \right) + 9 \]  
\[ 2.691 = \frac{2\pi}{15} (t - 11.25) \]  
\[ t = 17.673 + 15n \]

\[ -9 = 10 \cos \left( \frac{2\pi}{15} (t - 11.25) \right) \]  
\[ 6.424 = t - 11.25 \]  
\[ t = 17.673 + 15n \]

\[ -\frac{9}{10} \]  
\[ \cos^{-1} \left( \frac{-9}{10} \right) = \frac{2\pi}{15} (t - 11.25) \]  
\[ t = 2.673 + 15n \]

Between 2.673 minutes and 4.827 minutes.
8. **Ferris Wheel Problem:** As you ride the Ferris wheel, your distance from the ground varies sinusoidally with time. The last seat is filled, the Ferris wheel starts, and you start a stopwatch. After 20 seconds, you arrive at the top of the Ferris wheel, which makes a revolution once every 340 seconds. The diameter of the wheel is 64 feet and the bottom of the wheel is 3 feet above the ground.

a) Sketch a graph of this sinusoidal function.

![Graph of a sinusoidal function]

b) Write an equation expressing your height on the Ferris wheel as a function of time.

\[ y = 32 \cos \left( \frac{\pi}{170} (t - 20) \right) + 35 \]

(c) How high above the ground are you after 15 minutes?

15 minutes = 900 seconds

\[ y = 32 \cos \left( \frac{\pi}{170} (900 - 20) \right) + 35 \]

\[ y = 7.793 \text{ feet} \]

d) At what times will you be 32 feet above the ground?

\[ 32 = 32 \cos \left( \frac{\pi}{170} (t - 20) \right) + 35 \]

\[ -3 = 32 \cos \left( \frac{\pi}{170} (t - 20) \right) \]

\[ -\frac{3}{32} = \cos \left( \frac{\pi}{170} (t - 20) \right) \]

\[ \cos \left( -\frac{3}{32} \right) = \frac{\pi}{170} (t - 20) \]

\[ 1.665 = \frac{\pi}{170} (t - 20) \]

\[ -90.081 = t - 20 \]

\[ t = 110.081 + 340n \]

\[ t = 70.081 + 340n \]

\[ t = 369.919 + 340n \]

e) How high were you when the ride started?

\[ y = 32 \cos \left( \frac{\pi}{170} (0 - 20) \right) + 35 \]

\[ y = 64.839 \text{ feet} \]